JOURNAL OF COMPUTATIONAL PHYSICS 5, 463–486 (1970)

A Numerical Method for Calculating Steady Unsymmetrical Supersonic Flow Past Cones

MAURICE HOLT AND D. EJIKE NDEFO

Aeronautical Sciences Division, University of California, Berkeley, California

Received December 22, 1969

Telenin's numerical method is adapted to the problem of steady supersonic flow past pointed conical bodies at yaw. The method is formulated for cones of circular cross-sections with the intention of determining bounded analytic solutions uniformly valid in the region between the shock and cone surfaces. Attention is focused on the nature of the entropy field and the behavior of the streamlines as influenced by variations in the free-stream conditions.

1. INTRODUCTION

In the past decade, extensive theoretical investigations have been made in the area of steady unsymmetrical supersonic flow past pointed conical bodies. For such flows, the existence of a shock wave attached to the cone apex is guaranteed if the free-stream velocity is high enough and if the semiapex angle of the cone is not too large. Restriction of the motion between the shock and cone surfaces to the supersonic range gives rise to a class of flows characterized by the absence of a length scale–the class of *conical flows*. These flows are governed by an elliptic system of nonlinear differential equations in two independent variables. Because it is impossible to obtain closed-form analytical solutions to such nonlinear systems without making excessive simplifying assumptions, investigators have leaned heavily on numerical methods. With the renewed interest in supersonic flight and with the introduction of faster electronic computers, it becomes necessary to evolve simpler and more accurate numerical methods. Such methods should be able to provide not just the general solutions for conical flows but also detailed solutions in the important regions near the body surface. In addition, they should be valid for a wide range of angles of attack, free-stream Mach numbers and cone angles.

The first attempt to obtain nonlinearized numerical solutions of the problem was made by Stone [1, 2]. For supersonic flow past yawed cones with circular cross sections, he assumed that the solutions of the governing equations can be expanded as power series in the angle of attack, α. The first-order theory (1948) neglects second and higher powers of α. In this case, the differences between the values of field variables in yawed and unyawed flows are shown to be the first terms of Fourier sine or cosine series in the azimuthal angle ϕ. The coefficients of these series satisfy certain ordinary differential equations. Stone's solutions, extensively tabulated by Kopal [3], are however not uniformly valid at the cone surface and its immediate neighborhood. The dependence of pressure and density on Fourier cosine series implies that entropy on the cone surface is periodic with respect to the azimuthal angle. This is inconsistent with the fact that the cone surface is a stream surface on which entropy must be constant. The second-order theory (1952), which took into account second powers of α, improved Stone's results but did not remedy the above inconsistency.

The failure of Stone's theory to predict a constant entropy on the cone surface was first pointed out by Ferri [4]. He introduced the concept of the *entropy layer*—a layer of infinitesimal thickness next to the cone surface across which entropy changes rapidly from the value predicted by Stone's theory to the constant value on the cone surface. In the same paper, Ferri proved the existence of entropy singularities in unsymmetrical flows past cones. At those *vortical singularities* entropy is multivalued. Thus, in the region between the cone and shock surfaces, constant entropy surfaces would originate at the shock. On entering the entropy layer, they turn sharply and converge at the vortical singularities. For circular cones at small angles of attack, only one such singularity was shown to exist—being located at the intersection of the plane of symmetry and the leeward generator of the cone. Ferri further suggested that for large angles of attack this vortical singularity *lifts off* the cone surface to a new position in the plane of symmetry.

Following Ferri's entropy correction on Stone's theory, a variety of studies of the problem have been conducted by many authors. Common to all these later studies is an attempt, through varied numerical approaches, to verify and extend Ferri's results. In this respect, attention was brought to focus on three aspects of the problem:

(i) the shock shape and the nature of the streamlines in the region outside the immediate neighborhood of the cone surface,

(ii) the thickness of the entropy layer as influenced by variation of the angle of attack and the free-stream Mach number, and

(iii) the behavior of streamlines near the vortical singularities and verification of any *lift off* of these singularities.

Of the numerous numerical methods tried out on conical flow problems, the following are outstanding: the *inverse method*, Dorodnitsyn's method of integral relations, the BVLR time-limiting method, and a combination of matched asymptotic expansions and the PLK method. In the inverse method (Radhakrishnan [5], Briggs [6, 7], Mauger [8], Stocker and Mauger [9], a shock wave configuration is prescribed and numerical integration used to determine the hypothetical body shape producing the shock. Using a desk calculator, Radhakrishnan [5] looked for the body shape which produces a circular shock wave of semiapex angle 30° yawed at an angle of 20° with respect to a free stream of Mach number 10. Mauger [8] based his inverse method on the Garabedian technique of analytic continuation in a complex plane. He integrated along two families of characteristics starting from prescribed shocks of elliptic cross sections. Using two stream functions as independent variables and iterating on assumed shock configurations, Stocker and Mauger [9] obtained solutions for the direct problem of supersonic flow over a circular cone of semiapex angle 20° yawed at angles 5°, 10°, and 15° to a free stream of Mach number 3.53.

In all computations employing the inverse method, fairly accurate results were obtained for cases in which the angles of attack are small. For these cases (Stocker and Mauger [9]), the streamlines converge on the vortical singularities with an envelope-like behavior around the hypothetical body. The envelope-like behavior makes it easy to determine the body shape especially where the integration is carried out along streamlines. However, the high gradients of flow variables in the entropy layer makes it impossible to determine with sufficient accuracy the values of these variables on the cone surface. For moderate to high angles of attack, inverse method computations are beset with numerous difficulties. Radhakrishnan [5] encountered difficulties at the leeward side of the cone, resulting in a body shape with a *hump* on that side. The streamlines, while showing a tendency to converge, were irregular and blurred near the hump. Such results can either be attributed to the low accuracy inherent in the type of calculating machine used or to the inability of the inverse method to predict the nature of the flow in the crucial region surrounding the vortical singularity. However, Stocker and Mauger [9] recomputed the same problem using a more sophisticated machine and also obtained poorly defined body shape and wild streamlines. In the results by Stocker and Mauger for the flow over a circular cone at 10° angle of attack, the unexplained *hump* showed up again. The situation was worse for the 15° angle of attack and the authors could only conjecture that the phenomenon might be due to the *lift off* of the vortical singularity at high angles of attack as suggested by Ferri.

Based on the highly successful application of the method of integral relations to blunt body problems (Van Dyke, 1958; Belotserkovskii, 1966), it had been anticipated that the method would also solve conical flow problems. However, Chushkin and Shchennikov [10], Brook [11], Melnik [12] and Ndefo [13] have all proved the method to be unsuitable. Brook and Melnik experienced great numerical difficulties in attempting to solve the problem of supersonic and hypersonic flows past elliptic cones. Using the first approximation of the method of integral relations, Chushkin and Shchennikov obtained solutions for the often computed case of supersonic flow past a cone of semiapex angle 20° yawed at 5° to a free-stream of Mach number 3.53. No details of numerical computations or hint of any difficulties were given. Moreover, the first approximation determines only the shock shape and the values of the flow variables at the cone surface. It provides no information on the nature of the streamlines, and the existence of entropy layer and vortical singularity. The same case was investigated by Ndefo [13] who noted that, while results can be obtained to sufficient accuracy, the amount of computer time required and the numerical difficulties involved made it impossible to extend the method of integral relations to the necessary higher approximations.

The BVLR time-limiting method was proposed by the Russian school of mathematicians (Babenko et al. [14]). In this method, the direction of the axis of the cone is injected into the equations as an extra independent variable t with a time-like behavior. The procedure transforms the elliptic differential equations of conical flow into a t-hyperbolic system with Cauchy data on the plane $t = t_0 =$ constant, and boundary conditions along the body and at an unknown shock wave front. The solutions of conical flow are then obtained as the limit of solutions of the new system when $t \to \infty$. An algorithm is formulated in terms of non-linear implicit scheme and an iterative process used to linearize and make the scheme explicit. Although no rigorous proofs have been afforded about the existence of the limit, this method has been used to obtain good solutions for a wide range of angles of attack and Mach numbers.

R. E. Melnik ([15]) used the method of matched asymptotic expansions about the known Taylor–Maccoll solutions for axisymetric conical flows to investigate the problem of supersonic flow around slightly yawed cones with nearly circular cross-sections. The entropy layer was regarded as the *inner region* of the expansion, while the *outer region* is the rest of the area between the shock and cone surfaces. PLK coordinate-straining method is used to remove moving singular points that occur in the analysis of the inner region. First-order solutions for an elliptic cone at zero incidence, and the first-and second-order solutions for a circular cone at small incidence are obtained. Melnik, however, noted that while his solutions are uniformly valid in the entropy layer, they are not valid at the vortical singularity itself. No lift off of the vortical singularity was observed although a proof that such a behavior is possible under certain conditions was given. Melnik [15, 16] also carried out a comprehensive analysis of the nature of the streamline patterns in conical flows for various angles of attack, with particular attention on the behavior of the streamlines in the neighborhood of the vortical singularity. Holt [17], Cheng [18, 19], Woods [20], and Munson [21] also conducted such analyses.

Since the present paper was completed, three other calculations have been brought to the authors' attention. Bausset [22] developed a solution for unsymmetrical flow past a cone by the Pade Shanks method and applied this to both circular and elliptic cones; his results compare well with the BVLR method. Jones and South et al. [24] have developed methods for integrating between the shock and the cone surface along meridian planes. These are in the same spirit as the method developed here, but use different representations of the unknowns in the circumferential direction.

We conclude this survey of previous investigations by mentioning the experimental work of Holt and Blackie [25] which will form one of the bases for comparison with results obtained by the present investigation.

In the present investigation, we present a numerical method which, for a wide range of free-stream conditions, gives valid solutions throughout the flow region between the shock and cone surfaces. Attention will be focused on the nature of the entropy field and streamline patterns outside and inside the entropy layer with the intention of verifying how variations in free-stream conditions influence the behavior of the streamlines near the vortical singularity.

It is well known that Cauchy's problem is, in general, improperly posed for an elliptic system of equations. Yet it would be desirable to exploit the obvious numerical advantages of working with Cauchy-type problems. For an a priori restricted class of solutions (such as the class of bounded analytic functions), Cauchy's problem becomes correctly posed for elliptic systems. In this investigation, we seek a class of bounded analytic solutions to the elliptic system of differential equations governing conical flows. It is then possible to formulate a numerical method for a Cauchy-type problem with initial data prescribed at the shock surface. Based on the similarity of our problem to that posed in the subsonic region of supersonic blunt body flows, we adapt the numerical method first proposed by Telenin for axisymmetric blunt body problems (Gilinskii et al. [26]).

Without loss of generality, our attention will be confined to flows past cones of circular cross sections possessing one plane of symmetry on which the free-stream velocity vector lies. In this case, the pertinent region governed by conical flow equations is a spherical surface bounded by the circular profile of the cone and an unknown shock profile. Adapting Telenin's method, a number of meridional half-planes are passed through the flow region; intersecting the shock profile at nodes two of which lie in the symmetry plane. It is shown that the spherical flow region can be conveniently transformed to a plane rectangular domain by resorting to coordinates referenced with respect to the cone surface. Taking into account the periodicity of flow variables with respect to the azimuthal angle, their derivatives in this direction are approximated by the derivatives of trigonometric interpolation polynomials. The procedure reduces the system of partial differential equations to approximate ordinary differential equations holding simultaneously on the meridional half-planes. Approximate Cauchy data are assigned by prescribing at the nodes of the shock profile the constant coefficients of a trigonometric interpolation polynomial which defines an assumed shock shape. We can then integrate the approximate system numerically from the shock to the body surface. A stable scheme is formulated for iterating on the assumed shock shape so as to satisfy the boundary conditions at the cone surface.

As an example, numerical computations are carried out for the steady flow past a circular cone of semiapex angle 20°, yawed to a free stream of Mach number 3.53 at angles varying from 0 to 20°. Additional computations are made for the axisymmetric flow past the same cone at Mach number 5. The calculated results are compared with those based on other numerical methods and with the experimental results of Holt and Blackie [25].

2. Formulation of the Conical Flow Problem

General Equations and Boundary Conditions

The equations of motion for supersonic flow past a cone are first referred to a spherical coordinate system (r, θ, ϕ) such that $r = 0$ in the apex of the cone. $\theta = 0$ coincides with the cone axis and the meridian plane $\phi = 0$ coincides with the plane of symmetry on the windward side of the cone (Fig. 1). Velocity components are written in dimensionless form in terms of the critical speed of sound a^* while the density is divided by the density at infinity upstream ρ^* and the pressure by $\rho^* a^{*2}$.

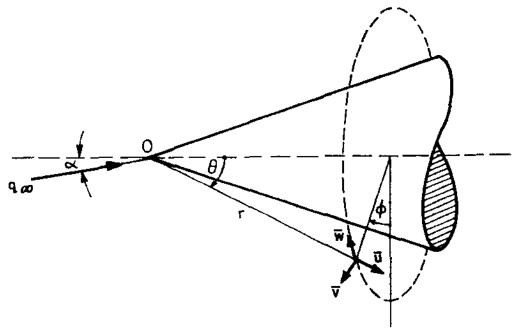

Fig. 1. Coordinate system and velocities.

We assume that the equation of state for a perfect gas is satisfied with constant specific heats γ. The problem can then be expressed in terms of five unknowns: the pressure, the density, and three velocity components. There must be solutions of the three momentum equations, the continuity equation and the condition that entropy is conserved on streamlines. Since the flow is conical, these solutions will be independent of r.

The boundary conditions are assigned at the intersections of the sphere r = constant, with the shock and cone surfaces. At the intersection with the circular cone surface, defined by $\theta = \theta_c =$ constant, the normal velocity component must be identically zero.

$$v = 0 \ (\theta = \theta_c). \tag{2.1}$$

At the intersection with the shock surface, defined by $\theta = \theta_s(\phi)$, the boundary conditions are the shock equations. These are derived, in terms of the free-stream conditions and shock shape, from the Rankine–Hugoniot relations,

$$u_s = q_\infty n_1 , \tag{2.2}$$

$$v_s = q_\infty n_3 \sin \epsilon \cos \epsilon - n_4 , \tag{2.3}$$

$$w_s = q_\infty n_3 \cos^2 \epsilon + n_4 \tan \epsilon, \tag{2.4}$$

$$\rho_s = q_\infty n_2 \cos^2 \epsilon / n_4 , \tag{2.5}$$

$$p_s = \left(\frac{2}{\gamma + 1}\right) q_\infty{}^2 n_2{}^2 \cos^2 \epsilon - \left(\frac{\gamma - 1}{2\gamma}\right) (1 - \mu^2 q_\infty{}^2), \tag{2.6}$$

where

$$n_1 = \cos \alpha \cos \theta_s - \sin \alpha \sin \theta_s \cos \phi, \tag{2.7}$$

$$n_2 = \cos \alpha \sin \theta_s + \sin \alpha \cos \theta_s \cos \phi + \sin \alpha \sin \phi \tan \epsilon, \tag{2.8}$$

$$n_3 = \frac{\sin \alpha \sin \phi - n_2 \sin \epsilon \cos \epsilon}{\cos^2 \epsilon}, \tag{2.9}$$

$$n_4 = \frac{1 - \mu^2 q_\infty{}^2 (1 - n_2{}^2 \cos^2 \epsilon)}{q_\infty n_2}, \tag{2.10}$$

$$\mu^2 = \frac{\gamma - 1}{\gamma + 1}, \tag{2.11}$$

$$\epsilon = \tan^{-1} \left(\frac{1}{\sin \theta_s} \frac{d\theta_s}{d\phi}\right), \tag{2.12}$$

$$q_\infty{}^2 = \frac{M_\infty{}^2}{\mu^2 M_\infty{}^2 + 2/(\gamma + 1)} . \tag{2.13}$$

ϵ is the angle, at any point on the shock surface, between the inward directed normal to the surface and the meridian plane $\phi =$ constant passing through that point.

Because of the presence of a plane of symmetry, it suffices to solve the problem in the half-plane $0 \leqslant \phi \leqslant \pi$. Additional conditions must however be imposed at the symmetry plane.

These are

$$w = 0 \qquad (\phi = 0, \pi) \tag{2.14}$$

and

$$\frac{d\theta_s}{d\phi} = 0 \qquad (\phi = 0, \pi). \tag{2.15}$$

Entropy Field and the Streamline Equation

Whenever the angle of attack is nonzero, the entropy jump across the shock wave front is no longer constant but varies according to the inclination of the shock to the free stream direction. As a consequence, the flow between the shock and cone surfaces is rotational. That entropy is conserved along a family of curves in the θ, ϕ plane is clearly indicated by the entropy equation in conical flow. These curves, which are the intersections of stream surfaces of constant entropy with the sphere r = constant, will henceforth be referred to as *streamlines*. Their equations are

$$\left(\frac{d\theta}{d\phi}\right)_L = \frac{v \sin \theta}{w}. \tag{2.16}$$

In the disturbed region between the shock and the cone surfaces Eq. (2.16) gives the tangent to the streamline at any point on the sphere r = constant. The streamlines emanate at the shock surface at angles which vary with ϕ. Substitution of Eqs. (2.1) and (2.14) into Eq. (2.16) shows that the intersections of the cone surface and the meridian planes $\phi = 0$, $\phi = \pi$ with the sphere r = constant are streamlines. Since streamlines cannot intersect at nonsingular points, it follows that the streamlines originating at the shock surface can neither terminate at the cone surface nor at the meridian planes of symmetry unless entropy singularities exist there. The entropy equation and Eq. (2.16) indicate that these *vortical singularities* occur at points where both v and w are zero. The above development was due to Ferri [4]. As has been noted in the introduction, only one such singularity exists for a circular cone.

Transformation of Coordinates

It will be advantageous to transform the spherical polar coordinates (r, θ, ϕ) into the coordinates (r, ξ, η) referenced with respect to the cone surface

$$\eta = \frac{\phi}{\pi}, \tag{2.17}$$

$$\xi = \frac{\theta_s(\eta) - \theta}{\sigma(\eta)}, \tag{2.18}$$

where

$$\sigma(\eta) = \theta_s(\eta) - \theta_c. \tag{2.19}$$

The new coordinate system makes it easier to extend the present method to cases where the cone has noncircular cross sections. In addition, the transformation makes it possible to evolve an equal interval difference scheme in the ξ direction for the integration of the differential equations of motion.

Equations (2.17) and (2.18) convert the region of interest $0 \leqslant \phi \leqslant \pi$, $\theta_c \leqslant \theta \leqslant \theta_s$ into the square domain $0 \leqslant \eta \leqslant 1$, $0 \leqslant \xi \leqslant 1$ (Fig. 2).

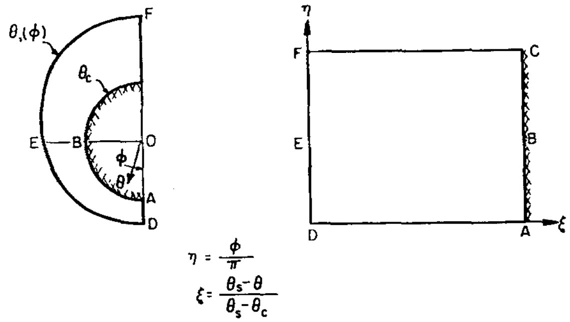

FIG. 2. Transformation of the flow region.

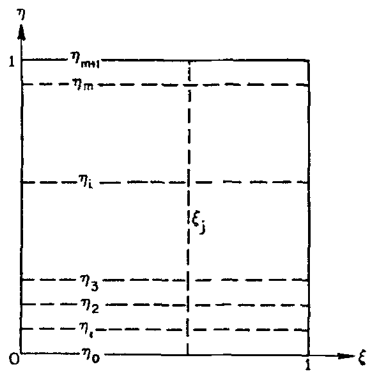

FIG. 3. Partitioning of the flow region.

3. TELENIN'S METHOD

Mathematical Basis of the Method

In problems of supersonic flow around blunt bodies which give rise to detached shock waves, the differential equations governing the flow in the subsonic region are elliptic. While network difference schemes do provide solutions to these problems, such schemes require computers with large memory and involve lengthy computation. In contrast, numerical methods designed for Cauchy-type problems require less memory and are much faster. Thus, if a numerical scheme can be formulated in terms of Cauchy's problem for elliptic equations, it should be superior to the traditional network difference schemes. However, Cauchy's problem for elliptic equations and systems is improperly posed (in the sense that small perturbations of initial data can induce large perturbations of the solution).

The mathematical basis of the Telenin Method (Gilinskii et al. [26]) rests on the fact that Cauchy's problem becomes correctly posed for elliptic equations if the region is bounded and a priori restrictions imposed on the class of solutions considered. One such class is the class of bounded analytic functions. Telenin then proceeded to show that one can formulate an approximating technique which, by adequately taking into account the analyticity of the sought-for solutions, provides a converging and sufficiently stable method for the numerical solution of the problem.

A Model for The Algorithm

As a model, Telenin chose the two-dimensional Laplace equation,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{3.1}$$

defined in the rectangular domain,

$$-1 \leqslant x \leqslant 1, \qquad 0 \leqslant y \leqslant b,$$

and subject to the Cauchy data,

$$\psi(x, 0) = f(x)$$

and (3.2)

$$\psi_y(x, 0) = g(x).$$

$(2n + 1)$ lines $x = x_j$ are drawn through the domain, intersecting the initial line $(y = 0; -1 \leqslant x \leqslant 1)$ in $(2n + 1)$ nodes. The desired solution is approximated by Lagrange interpolation polynomials for x,

$$\psi(x, y) \approx \psi_n^*(x, y) = \sum_{j=0}^{2n} \psi_{jn}^0(y)\, x^j, \tag{3.3}$$

where for any fixed $y = y_k$, $\psi^0{}_{jn}(y)$ is a linear function of the values $\psi_n^*(x_j, y)$ of the approximate solution at the $(2n + 1)$ nodes. On substituting Eq. (3.3) into Eq. (3.1) and imposing the requirement that the derived expression be satisfied identically on all the lines $x = x_j$, an approximating system of ordinary differential equations results.

$$\frac{d^2\psi_{kn}^*}{dy^2} + \sum_{j=2}^{2n} j(j-1)\,\psi_{jn}^0(y)\,x_k^{j-2} = 0, \qquad (k = 0, 1, 2, \ldots, 2n+1), \tag{3.4}$$

with initial conditions,

$$\psi_{kn}^*(0) = f(x_k), \qquad \psi_{kn_y}(0) = g(x_k). \tag{3.5}$$

This system can now be numerically integrated starting at the initial line and proceeding towards $y = b$. Telenin demonstrated that

$$\psi_{kn}^*(y) = \psi_n{}^*(x_k, y) = \operatorname{Re}[\lambda + i\zeta], \tag{3.6}$$

where

$$\lambda = \sum_{j=0}^{2n} a_j{}^+ z_k{}^j, \qquad \zeta = -\sum_{j=0}^{2n} \frac{b_j{}^+}{j+1} z_k^{j+1}, \tag{3.7}$$

and

$$z_k = x_k + iy$$

is a solution of Eq. (3.4). This solution converges as $n \to \infty$, the dimensions of the region of convergence depending on the position of the singularities of the initial data.

A Note on the Accuracy of the Method

Based on the analysis of the above model, the accuracy of the method is found to increase with the number of integration steps as a result of increase in the informa- [redacted] assumed in the beginning. The best number of lines $x = x_j$ that should be used depends on the desired accuracy of the approximation of the solution in the direction of the initial line. This in turn depends on the character of the specific problem being considered.

There is only one crucial restriction to the use of Telenin's method. The round-off errors during computation must be quite small before the method can form the basis for the solution of an elliptic system as a Cauchy-type problem.

4. Formulation of Numerical Algorithm

Telenin's Method Adapted to Conical Flows

The equations governing conical flows under investigation are elliptic in the two independent variables ξ, η. Here, ellipticity is defined by $(v^2 + w^2) < a^2$. We seek a class of analytic bounded solutions u, v, w, p, ρ to the set of conical equations subject to the boundary conditions (2.1) to (2.15). Based on the analysis of the last section, Cauchy's problem is correctly posed for this set, and accordingly Telenin's method can be used.

Through the flow region between the cone and shock surfaces we pass $(2n + 2)$ equally spaced meridional half-planes $\phi =$ constant, two of which coincide with the symmetry plane (i.e., $\phi = 0$, $\phi = \pi$). The intersections of these planes with the sphere $r =$ constant are $(m + 2)$ lines $\eta = \eta_i =$ constant in the transformed plane of interest $0 \leqslant \eta \leqslant 1$, $0 \leqslant \xi \leqslant 1$ (Fig. 3).

If the shock profile $\xi = 0$ is regarded as the initial line, then the dependent variables and the shock angle should be approximated by polynomials in ξ. w is odd with respect to the symmetry plane, while u, v, p, ρ, θ_s are even. However, since the approximated functions are periodic with respect to $\phi = \eta\pi$, it will be more appropriate to use trignometric interpolation polynomials, namely,

$$\begin{pmatrix} u(\xi, \eta) \\ v(\xi, \eta) \\ p(\xi, \eta) \\ \rho(\xi, \eta) \end{pmatrix} \approx \sum_{k=0}^{m+1} \begin{pmatrix} u_k{}^0(\xi) \\ v_k{}^0(\xi) \\ p_k{}^0(\xi) \\ \rho_k{}^0(\xi) \end{pmatrix} \cos(k\eta\pi), \tag{4.1}$$

$$\theta_s(\eta) \approx \sum_{k=0}^{m+1} \theta_{s_k}^0 \cos(k\eta\pi), \tag{4.2}$$

and

$$w(\xi, \eta) \approx \sum_{k=0}^{m+1} w_k{}^0(\xi) \sin(k\eta\pi). \tag{4.3}$$

This latter approach has two advantages. The approximations are smoother and the conditions at the plane of symmetry are satisfied identically.

These expressions are substituted into the governing partial differential equations with the requirement that the resulting equations be satisfied identically on each line η_i. An approximating system of $5(m + 2)$ first-order ordinary differential equations is then obtained for the approximate values u_i, v_i, p_i, ρ_i of the dependent variables on the $(m + 2)$ lines $\eta_i =$ constant. For example, the equation for u_i is

$$\frac{du_i}{d\xi} = \frac{(v_i^2 + w_i^2)\, \pi \sin \beta_i - u_i' w_i}{\pi K_i \sin \beta_i}, \tag{4.4}$$

where

$$\beta_i = \theta_{s_i} - \xi\sigma_i$$

and K_i is a given function of the unknowns. The boundary conditions remain as given in Eqs. (2.1) to (2.13) with the subscript i appended to all variables.

5. Numerical Computations

General Remarks

To integrate the ordinary differential equations of type (4.1) numerically, constant coefficients $\theta^0_{s_k}$ are assigned at the $(m + 2)$ nodes of the initial line $\xi = 0$. The discrete values θ_{s_i} defining the shock shape can then be determined. Substitution of these into the shock relations (2.2) to (2.13) gives the initial values of the dependent variables for some prescribed angle of attack, free-stream conditions and cone angle.

The equations are now integrated step-by-step towards the cone surface $\xi = 1$, using a stable numerical scheme. At each step, the calculated values of the dependent variables must be used to determine the coefficients of the trigonometric polynomials. The coefficients are needed for the approximation of the derivatives of the variables with respect to η at this step. For Telenin's method to be valid during the process of integration, any numerical scheme chosen must incorporate the above procedure.

At the intersection of the lines $\eta_i =$ constant with the cone surface the boundary conditions $v_i = 0$ are tested. In general, these will not be satisfied by the first choice of $\theta^0_{s_k}$ at the initial line $\xi = 0$. Thus, a suitable iteration scheme has to be formulated for choosing the $\theta^0_{s_k}$ that will result in the satisfaction of the boundary conditions at the cone surface.

We now define a new parameter, Λ_i, by

$$\Lambda_i(S) = p_i\rho_i^{-\gamma}. \tag{5.1}$$

Since $\Lambda_i =$ constant along streamlines, it will, henceforth, be referred to as the *entropy parameter*. Λ_i plays two very important roles:

(i) Since the thin entropy layer surrounding the cone surface is characterized by very high gradients in entropy, the behavior of $d\Lambda_i/d\xi$ near the cone surface should indicate roughly the extent of the entropy layer,

(ii) The high gradient in entropy in the vortical layer is the result of a high gradient in the dependent variables. Unless a sufficiently small step-size is used near the cone surface, there may be large distortions in the derivatives of these variables with respect to the geometric coordinate ξ. Under these conditions, we may have a situation in which all $v_i \approx 0$ but $\Lambda_i \neq$ constant. This implies either that we are on a *false* cone surface or that the distortions have made the computed values on the cone unreliable. Thus, the requirement that $\Lambda_i \approx$ constant on $\xi = 1$ simultaneously with $v_i \approx 0$ serves as a guide in choosing step-size near the cone surface.

Scope of Numerical Work

Assuming $\gamma = 1.405$, the following cases are investigated for a circular cone of $\theta_c = 20°$.

Case I (Axisymmetric Flows).

(i) $M_\infty = 3.53$.
(ii) $M_\infty = 5$.

Case II (Unsymmetrical Flows).

(i) $M_\infty = 3.53, \quad \alpha = 5°$.
(ii) $M_\infty = 3.53, \quad \alpha = 10°$.
(iii) $M_\infty = 3.53, \quad \alpha = 15°$.
(iv) $M_\infty = 3.53, \quad \alpha = 20°$.

These cases were selected because experimental results and theoretical computations based on assorted numerical methods are available for comparison. The two most common bases for comparison are the pressure coefficient at the cone surface and the integrated lift and drag coefficients. For this analysis the pressure coefficient is given by

$$C_p = \frac{2}{M_\infty} \left[\frac{(1 + (\gamma - 1)/2)\, M_\infty^2)}{(\gamma + 1)/2}\, p - \frac{1}{\gamma} \right]. \tag{5.2}$$

In all the cases investigated, an 8-point approximation ($2m + 2 = 8$) is used. This implies 8 points on the shock surface or equivalently 5 lines $\eta_i =$ constant through the relevant region $0 \leqslant \eta \leqslant 1$, $0 \leqslant \xi \leqslant 1$. A total number of 25 coupled first-order ordinary differential equations then result.

A fifth-order Runge–Kutta scheme with automatic error and step-size controls is used to integrate the equations on the CDC 6400 computer. This scheme, originally derived by Zonneveld [27] and later modified by Downton [28], uses six intermediate points in each interval and one additional point for step control. Thus, it performs seven derivative evaluations at each step. It uses variable step size in order to achieve a given accuracy by a minimum number of steps. In this investigation, the relative error bound was preset at 10^{-5}.

6. Results and Discussion

A Note on Axisymmetric Cases

In the case of axisymmetric flows, the nondependence of the flow variables on the azimuthal angle ϕ lowers the order of the equations by one. Accordingly, the extent of nonlinearity of the equations and the boundary conditions are reduced. Because of these reasons, computations made for axisymmetric flows can be regarded as some sort of test cases for checking the accuracy and validity of the Telenin method. The tests are, however, informal since no entropy layers or vortical singularities exist in axisymmetric flow.

Computed results are tabulated in Appendix C. Compared with the results

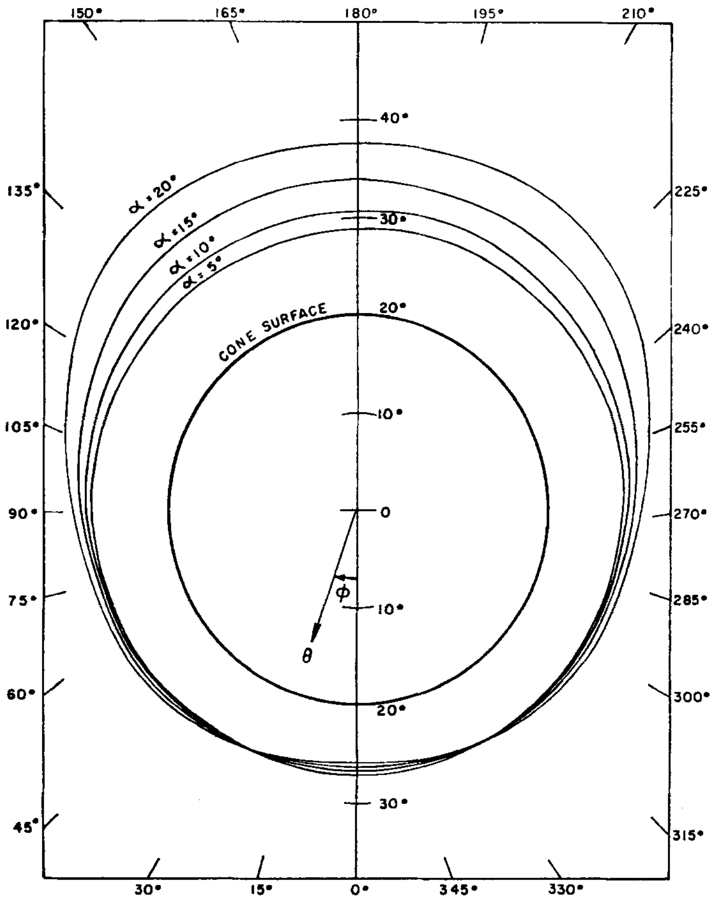

FIG. 4. Dependence of shape of shock wave on angle of attack.

obtained by the Ames Research Staff [29], the present method gives shock shapes, θ_s, with maximum error of 0.5 %.

The convergence of the numerical scheme for these test cases was found to be very fast.

Shape of Shock Wave Front

Figure 4 gives the shock profiles on a sphere r = constant for various angles of attack. None of the profiles are circular but those for 5 and 10° angles of attack are nearly circular. The figure seems to indicate that all the profiles do intersect at a common point. The variations in shock angle at the windward plane of symmetry as the angle of attack increases occur in the third decimal place of radian measure. In contrast, the variations at the leeward side are quite substantial.

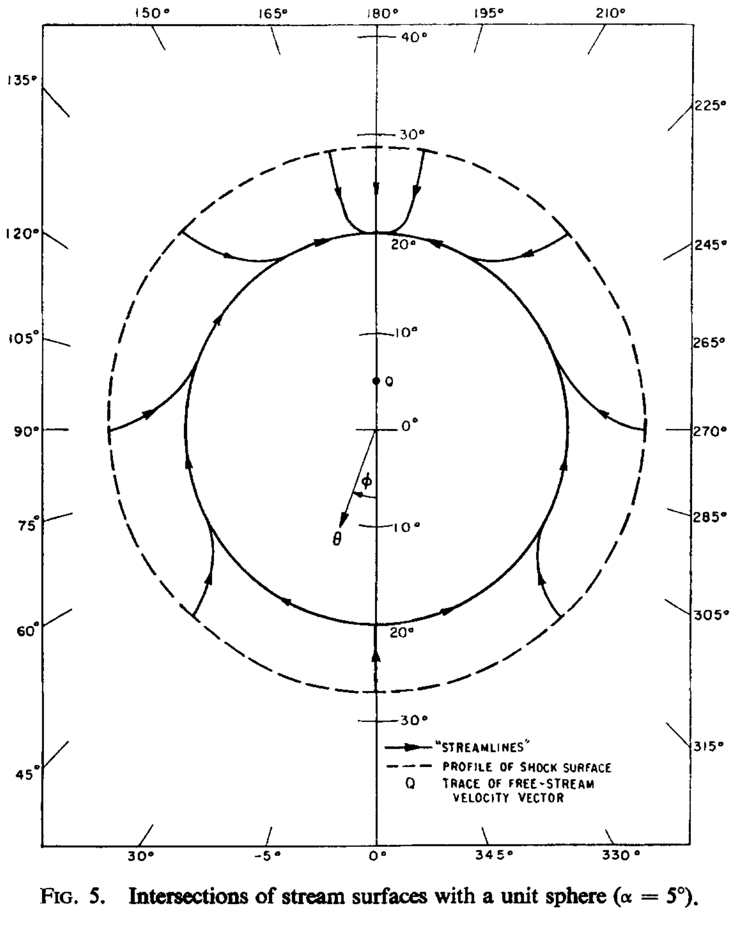

FIG. 5. Intersections of stream surfaces with a unit sphere (α = 5°).

Entropy Field and Streamline Patterns

The streamline patterns for various angles of attack are shown in Figs. 5–8. They indicate the nature of the entropy field whose actual behavior is better characterized by the plot of the entropy parameter, $\Lambda = p/\rho^{\gamma}$ in Fig. 9.

of the plane of symmetry with the cone surface is confirmed. The stagnation point at the windward side behaves like a saddle point with respect to the streamlines; while the stagnation point at the leeward side is a node to which all the streamlines converge. The existence of the vortical singularity is thus confirmed. However, within the range of angles of attack covered by the present investigation, no evidence was found of the lift-off of the vortical singularity as suggested by Ferri.

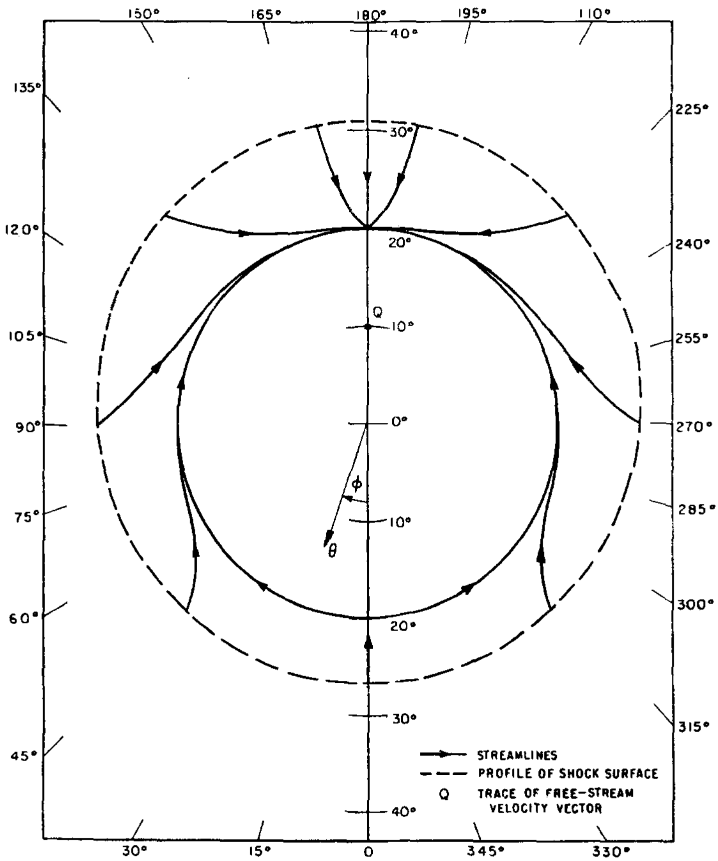

FIG. 6. Intersections of stream surfaces with a unit sphere ($\alpha = 10°$).

For moderate to high angles of attack, various theoretical and experimental results have shown that the minimum of surface pressure distribution shifts from the leeward plane of symmetry to some point between $\phi = 0$ and $\phi = \pi$. R. E. Melnik [16] has suggested that under this condition the vortical singularity may be displaced to the new minimum of surface pressure. Our computations for 20° angle of attack confirm the shift in the minimum of surface pressure (Fig. 10), but there is no evidence of the displacement of the vortical singularity (Fig. 8) as suggested by Melnik.

At 5° angle of attack all the streamlines appear to touch the cone surface before they reach the vortical singularity, thus exhibiting an *envelope-like* behavior observed by Stocker and Mauger [9]. As the angle of attack increases, this behavior becomes gradually lost; with the result that streamlines appear as if they are being blown away from the cone face. For example, at 10° angle of attack, the

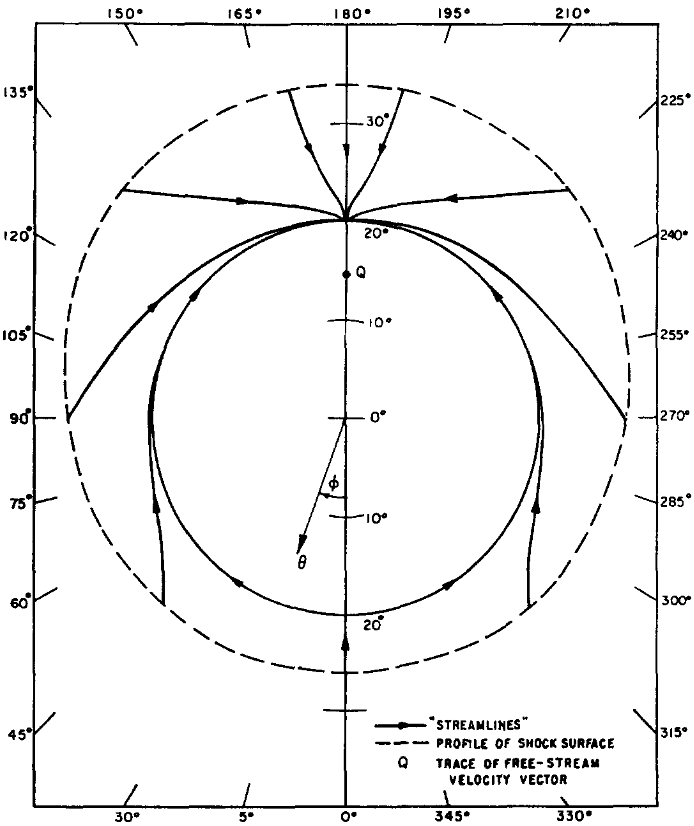

FIG. 7. Intersections of stream surfaces with a unit sphere ($\alpha = 15°$).

streamlines emanating from the shock at points in the interval $3\pi/4 \leqslant \phi \leqslant \pi$ do not touch the cone surface before reaching the vortical singularity. At 15° angle of attack, the interval increases to approximately $\pi/2 \leqslant \phi \leqslant \pi$. The relation between this phenomenon and the direction from which the streamlines enter the vortical singularity is obvious. At small angles of attack, all the streamlines enter the singularity from a direction tangential to the cone surface; at high angles of attack, the direction becomes normal to the cone surface (i.e., tangential to the plane of symmetry); at intermediate angles of attack, the directions can range from tangential to normal to the cone surface.

The existence of the entropy layer is confirmed by the behavior of the streamlines near the cone surface especially at 5° angle of attack. The high gradients of

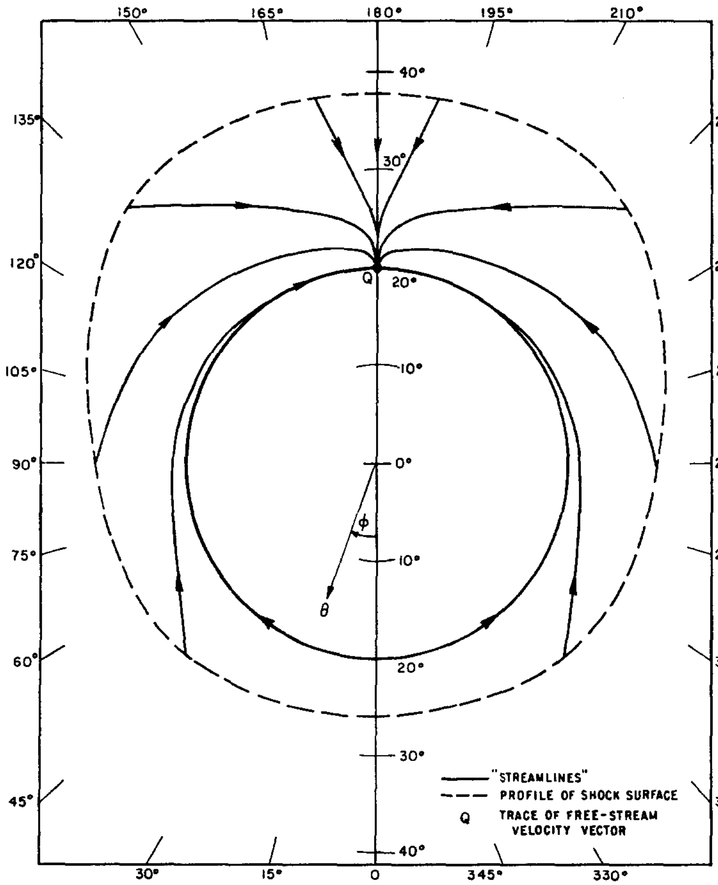

FIG. 8. Intersections of stream surfaces with a unit sphere ($\alpha = 20°$).

entropy in the region ($\xi \approx 1$) are evident in the plots of entropy parameter for 5 and 10° angles of attack (Fig. 10). Two inferences can be made from this plot. Firstly, the gradients of entropy in the layer are steeper at the leeward side of the cone than at the windward side. This seems physically justifiable by the proximity of the leeward side to the vortical singularity. Secondly, in any given meridian plane, the gradients of entropy near the cone surface become less steep with increasing angle of attack. This suggests that, contrary to the findings of Stocker and Mauger, the entropy layer does not remain uniformly thin at all angles of attack. The thickness increases with the angle of attack. Our computations show that at 20° angle of attack the entropy layer is quite thick compared with the case of the 5° angle of attack. Melnik [15] obtained similar estimates of entropy layer thickness using the method of matched asymptotic expansions.

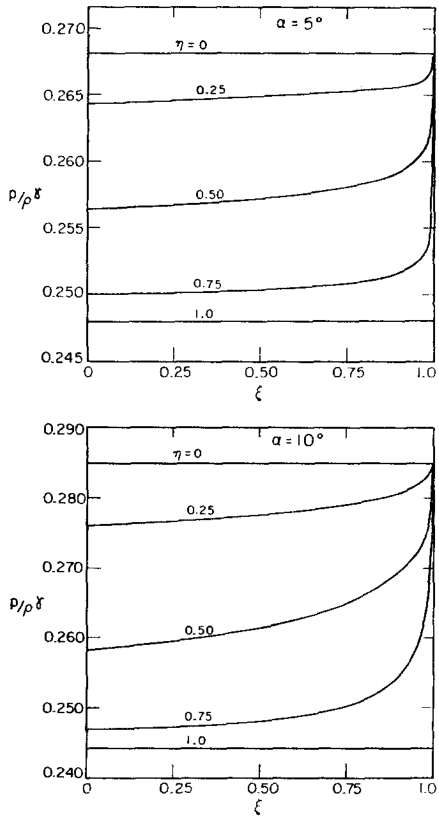

FIG. 9. Variation of entropy parameter, $\Lambda = p/\rho^{\gamma}$, along lines η = constant.

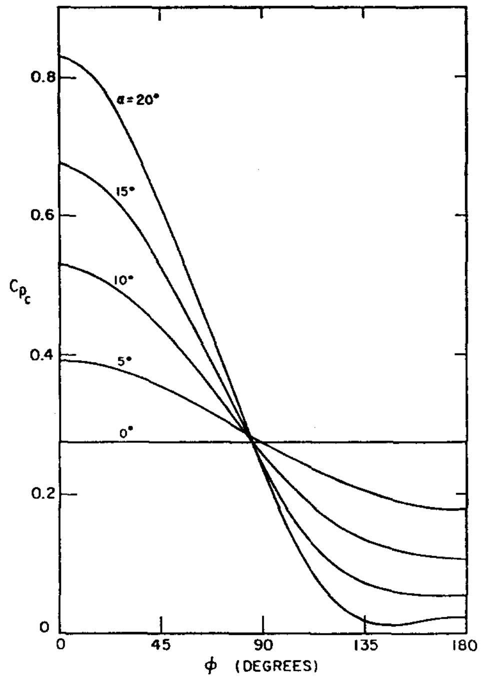

FIG. 10. Dependence of pressure coefficient at the cone surface on angle of attack.

Surface Pressure Coefficient: Integrated Drag and Lift Coefficients

The dependence of surface pressure coefficient on angle of attack is shown in Fig. 10. The extent of the accuracy which Telenin's method can achieve is indicated by comparisons with results of experiment and other numerical methods (Figs. 11, 12). Experimental values are from the work of Holt and Blackie [25]; while theoretical results are those obtained by the first approximation of the method of integral relations, and from the Kopal tables for Stone's first- and second-order theories.

In all cases investigated, the computed surface pressure coefficient and the integrated drag and lift coefficients compare very favorably with experimental values. The computed values of surface pressure coefficient are generally lower than the esperimental values near the leeward plane of symmetry; the maximum error being approximately 4.5 %. As the angle of attack increases, the computed values of integrated drag and lift coefficients drop below experimental values; a maximum error of 4 % occurring at $\alpha = 20°$.

In general, a closer agreement with experimental observation is obtained by the present method than by either the method of integral relations or Stone's first- and second-order theories. The only other numerical method which has been shown to produce as good an agreement with experimental results is the BVLR time-limiting method (Babenko et al. [14]).

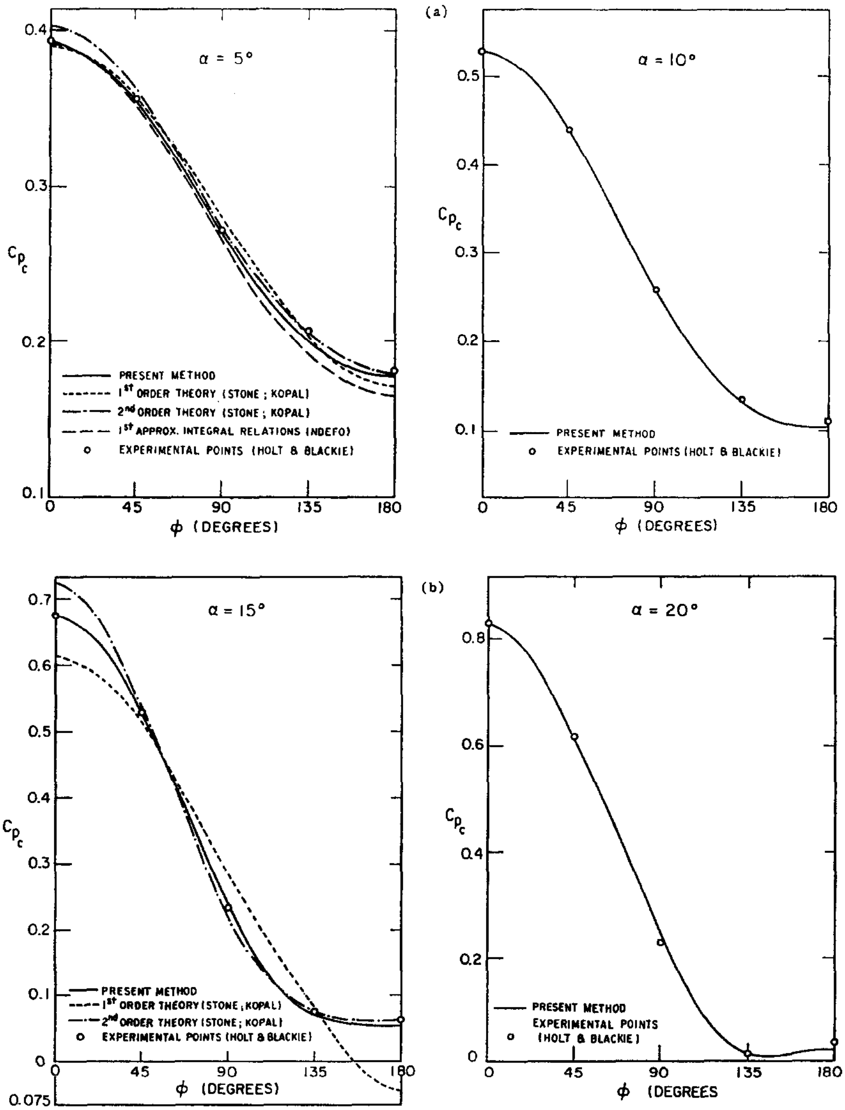

FIG. 11. (a) Comparison of theoretical and experimental values of pressure coefficient at the cone surface. (b) Comparison of theoretical and experimental values of pressure coefficient at the cone surface.

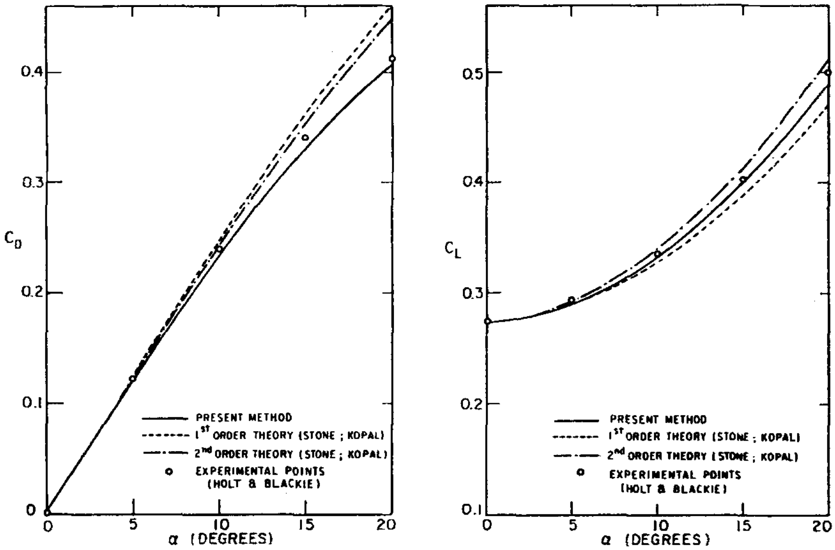

FIG. 12. Dependence of integrated drag and lift coefficients on angle of attack (experimental and theoretical comparison).

BIBLIOGRAPHY

K. I. BABENKO AND G. P. VOSKRESENSKII, *Zh. Vychisl. Mat. Mat. Fiz.* **1** (1961), 1051–1060.

O. M. BELOTSERKOVSKII, Computer Centre of the Academy of Sciences USSR, Moscow (NASA TT F-453, 1967).

O. M. BELOTSERKOVSKII AND P. I. CHUSHKIN, *Zh. Vychisl. Mat. Mat. Fiz.* **2** 5, 731–759.

L. BERS AND L. NIRENBERG, Convegno Internazionale sulle Equazioni Lineari alle Derivate Parzialli, Edizioni Cremonese, Rome, (1955), 141–167.

L. COLLATZ, The Numerical Treatment of Differential Equations, Springer-Verlag, New York, 1966.

C. FERRARI, NACA Tech. Memo. No. 1333 (1962).

A. FERRI, General Theory of High Speed Aerodynamics, Vol. VI of "Princeton Series in High Speed Aerodynamics and Jet Propulsion" (W. R. Sears, Ed.), Section 11, Chap. 3 (1954).

A. FERRI, N. NESS, AND T. T. KAPLITA, *J. Aero. Sci.* **20**, 8, 563–571 (1963).

P. GERMAIN, Tenth Lanchester Memorial Lecture, *J. Roy. Aero. Soc. London* **71**, 679–681 (1967).

J. W. MACCOLL, *Proc. Roy. Soc. A* **159**, 459–472 (1937).

R. E. MELNIK, Grumman Aircraft Engineering Corp., FDL-TDR-64-82 (1965).

R. C. ROBERTS AND J. D. RILEY, *J. Aero. Sci.* **21**, 5, 336–342 (1954).

G. I. TAYLOR AND J. W. MACCOLL, *Proc. Roy. Soc. A*, **139**, 278–311 (1933).

G. F. TELENIN AND G. P. TINYAKOV, *Dokl.* AN SSSR **154**, 5, 1056–1058 (1964).

M. D. VAN DYKE, *J. Aero/Space Sci.* **25**, 8, 485–496 (1958).

G. B. YOUNG AND C. P. SISKA, *J. Aero. Sci.* **19**, 111–119 (1952).

References

1. A. H. Stone, *J. Math. Phys.* **27** (1948), 67–81.
2. A. H. Stone, *J. Math. Phys.* **30** (1952), 200–213.
3. Z. Kopal, M.I.T. Dept. Electrical Engineering, Center of Analysis, Tech. Report No. 5, 1949.
4. A. Ferri, NACA TR 1045, 1951.
5. G. Radhakrishnan, "The Exact Flow Behind a Yawed Conical Shock." College of Aeronautics, Cranfield, England. Report No. 116, 1958.
6. B. R. Briggs, NASA TN D-24 (1959).
7. B. R. Briggs, NASA TN D-340 (1960).
8. F. E. Mauger, British Armament Research Establishment, Report (B) 3/60, 1960.
9. P. M. Stocker and F. E. Mauger, *J. Fluid Mech.* **13** (1962), 383–399.
10. P. I. Chushkin and V. V. Schchennikov, *AN SSSR Inzh. Fiz. Zh.* **3** (1962), 88–94 (Grumman Research Translation TR-20, 1962).
11. J. W. Brook Grumman Aircraft Engineering Corp. FDL-TDR-64-7 (1964).
12. W. L. Melnik, "Supersonic and Hypersonic Flow of an Ideal Gas About Elliptic Cones by Method of Integral Relations," Ph.D. Dissertation, University of Minnesota, 1964.
13. E. Ndefo, University of California, Berkeley, Aero. Sci. Div. Report AS-68-2, 1968.
14. K. I. Babenko, G. P. Voskresenskii, A. N. Lyubimov, and V. V. Rusanov, *AN SSSR Matem. Inst. im V. A. Steklova Moskova* (English transl. by Israel Program for Scientific Translations, 1968).
15. R. E. Melnik, "A Systematic Study of Some Singular Conical Flow Problems," Ph.D. Dissertation, Polytechnic Institute of Brooklyn, 1965.
16. R. E. Melnik, "Vortical Singularities in Conical Flow," Grumman Aircraft Engineering Corp., RE-251J, 1966.
17. M. Holt, *Quart. J. Mech. Appl. Math.* **7** (1954), 438–445.
18. H. K. Cheng, *J. Aero. Sci.* **27** (1960), 155–156.
19. H. K. Cheng, *J. Fluid Mech.* **12** (1962), 169–191.
20. B. A. Woods, *Aero. Quart.* **13** (1962), 115–128.
21. A. G. Munson, *J. Fluid Mech.* **20** (1964), 625–643.
22. M. Bausset, Étude par la Méthode de Padé-Shanks des Chocs Stationnaires Attachés au Sommet d'un Cone du Second Ordre, Inst. de Méc. des Fluides, Univ. D'Aix-Marseille, 1969.
23. D. J. Jones, Nat. Res. Council of Canada, Aero. Report No. LR-507, NRC No. 10361, Ottawa, Canada, 1968.
24. Jerry C. South, Jr. and E. Bernard Klunker, Proc. NASA Symposium on Analytical Methods in Aircraft Aerodynamics, 1969, NASA SP-228.
25. M. Holt and J. Blackie, *J. Aero. Sci.* **23** (1956), 931–936.
26. S. M. Gilinskii, G. F. Telenin, and G. P. Tinyakov, *Izv. AN SSSR Mekhan. Mashmostr.* **4** (1965), 9–28 (NASA TT F-297, 1965).
27. J. A. Zonneveld, Automatic numerical integration, *Math. Centrum Amsterdam Afd. Zuivere Wisk.* **8** (1964).
28. M. Downton, Univ. of California Computer Center, Berkeley, Tech. Report No. 7, 1967.
29. Ames Research Staff, NACA Report No. 1135 (1953).